

On the sideways expansion of relativistic non-spherical shocks and GRB afterglows

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ABSTRACT

Expansion of non-spherical relativistic blast waves is considered in the Kompaneets (the thin shell) approximation. We find that the relativistic motion effectively “freezes out” the lateral dynamics of the shock front: only extremely strongly collimated shocks, with the opening angles $\Delta\theta \leq 1/\Gamma^2$, show appreciable modification of profiles due to sideways expansion. For less collimated profiles the propagation is nearly ballistic; the sideways expansion of relativistic shock becomes important only when they become mildly relativistic.

1. Introduction

Dynamics and corresponding radiative signatures of non-spherical relativistic shocks remains an important unresolved issues in studies on Gamma Ray Bursts (GRBs). Since GRBs produce narrowly collimated outflows that evolve laterally, understanding the overall dynamics - both theoretical and in terms of agreement between different numerical results - is imperative to the interpretation of the broadband observations of GRBs Rhoads (1999); Frail et al. (2001).

Presently, there are two competing views on the lateral evolution of the relativistic outflows. Theoretically, it is typically argued that the lateral evolution of the flow proceeds with relativistic velocities (Piran 1999), (see also Wygoda et al. 2011). This view is contradicted by the results of numerical simulations that show very little lateral evolution in the relativistic regime Cannizzo et al. (2004); Zhang & MacFadyen (2009); Meliani & Keppens (2010); van Eerten & MacFadyen (2011).

In this Letter we argue that this disagreement results from the incorrect theoretical assumptions about the lateral evolution of the flow. What is important for the interpretation of observations is the evolution of a curved shock. Previously the lateral evolution of the non-spherical shocks was incorrectly treated as a free lateral expansion into vacuum (*e.g.*, Wygoda et al. 2011, Eq. 5).. The assumption of the lateral expansion with the sound speed results in a “gramophone-type” profiles and *exponential* slowing down of the ejecta. This has drastic implications for the underlying light curves (eg Kumar & Panaitescu 2000; Panaitescu & Kumar 2003). In fact the dynamics of the non-spherical shocks is more subtle; the correct treatment, as we argue below, is consistent with slow lateral evolution seen in numerical simulations.

Evolution of strong non-spherical shocks is a well studied problems in fluid dynamics. The two fundamental works that have laid the foundation for non-spherical (two-dimensional) shocks, due to Kompaneets (1960) and to Laumbach & Probstein (1969), were originally designed to treat strong shock waves in the non-isotropic medium. These two complimentary methods have been extensively applied in astrophysics to treat supernova explosions (Bisnovatyi-Kogan & Silich 1995) and non-isotropic winds (*e.g.*, Icke 1988). In the Kompaneets approximation the internal pressure of the gas is assumed to be constant. Then the Rankin-Hugonio conditions determine the normal velocity of the shock in the external inhomogeneous medium. A modification of the Kompaneets approximation - a thin or snowplow shell approximation - has also been used extensively (*e.g.*, Wiita 1978; Mac Low & McCray 1988; Bisnovatyi-Kogan et al. 1989). In a complimentary Laumbach-Probstein approach (Laumbach & Probstein 1969) the streamlines of the shocked material are assumed to be radial, thus neglecting the lateral pressure forces.

The relativistic generalization of the Kompaneets and the Laumbach-Probstein methods have been discussed by Shapiro (1979). Relativistic dynamics provide extra support for the thin shell method, since in the relativistic blast waves the shocked material is concentrated in even narrower region R/Γ^2 than in the non-relativistic Sedov solution. In addition, the limited causal connection (over the angle $\sim 1/\Gamma$) provides a justification for the Laumbach-Probstein method on the angle scale comparable to $1/\Gamma$. As has been pointed out by Shapiro (1979), the two methods - Kompaneets and Laumbach-Probstein - become very similar in the relativistic regime. This is due to the fact the in a relativistic quasi-spherical wave, the typical angle that a shock wave makes with the direction of the velocity is of the order $\alpha \sim 1/\Gamma^2$. Thus the post shock pressure along the shock differs only by one part in Γ^2 , so that both approximations of constant post-shock pressure and radial post-shock motion become equivalent. In some sense the propagation of strongly relativistic non-spherical shocks becomes trivial: relativistic kinematic effect freeze out the lateral dynamics of the flow so that different parts of the flow behave virtually independently.

2. Relativistic non-spherical shocks in the thin shell approximation

In this section we re-derive the relativistic Kompaneets equation Kompaneets (1960); Shapiro (1979) allowing for the arbitrary velocity of the shock and arbitrary (angle-dependent) luminosity and/or external density. Consider a shock propagating with a three-velocity V at an angle α to its normal. There are three generic rest frames in the problem: laboratory frame K , a frame where the shock is normal to the flow K_1 and a shock rest frame K_0 . A frame K_1 is related to the lab frame K by a Lorentz boost along y axis with a Lorentz factor $\Gamma_{\parallel} = 1/\sqrt{1 - V^2 \sin^2 \alpha}$. In K_1 the velocity of the shock is $V_1 = \Gamma_{\parallel} V \cos \alpha$ (along the x direction). Thus, $\Gamma_1^2 = 1/(1 - V_1^2) = \Gamma^2 \cos^2 \alpha + \sin^2 \alpha$ (the shock becomes non-relativistic when $\pi/2 - \alpha \sim 1/\Gamma$). In the frame K_0 V_1 and Γ_1 are the velocity and the Lorentz factor of the unshocked medium. In the lab frame the x component of the shocked velocity $V'_x = V'_1/(\Gamma_{\parallel}(1 + V_1 V \sin \alpha))$ generally has a complicated form, but simple relations can be obtained in the strongly relativistic limit (see below).

We introduce next an acceleration parameter K (Kompaneets 1960, Icke 1988) as a Lorentz factor of the shock in the K_1 frame

$$K = \Gamma_1^2 \quad (1)$$

The acceleration parameter can be expressed in terms of a ratio of a post shock pressure P' to the upfront density ρ . For relativistic strong shocks with the ratio of specific heats $\hat{\gamma} = 4/3$

$$K = \frac{2}{3} \frac{P'}{\rho} \quad (2)$$

while for non-relativistic shock with $\hat{\gamma} = 5/3$

$$V_1^2 = 1 - \frac{1}{K} = \frac{4}{3} \frac{P'}{\rho} \quad (3)$$

Expressing the relevant quantities in terms of K we find

$$\begin{aligned} \Gamma &= \sqrt{\frac{K - \sin^2 \alpha}{\cos^2 \alpha}} \approx \frac{\sqrt{K}}{\cos \alpha} \\ V &= \sqrt{\frac{K - 1}{K - \sin^2 \alpha}} \approx 1 - \frac{\cos^2 \alpha}{2K} \\ V_1 &= \sqrt{1 - \frac{1}{K}} \approx 1 - \frac{1}{2K} \\ V_1' &= \sqrt{1 - \frac{2}{K}} \approx 1 - \frac{1}{K} \\ \Gamma_{\parallel} &= \sqrt{\frac{K - \sin^2 \alpha}{K \cos^2 \alpha}} \approx \frac{1}{\cos \alpha} \\ V_x' &\approx \frac{\cos \alpha}{1 + \sin \alpha} \left(1 - \frac{(2 + \sin \alpha) \cot^2 \alpha / 2}{2K} \right), \end{aligned} \quad (4)$$

where the approximations assume strongly relativistic motion.

Finally, expressing V in terms of V_1 we find

$$V^2 = \frac{V_1^2}{V_1^2 \sin^2 \alpha + \cos^2 \alpha} = \frac{1 - 1/K}{1 - \sin^2 \alpha / K} \approx \begin{cases} \frac{V_1^2}{\cos^2 \alpha} & \text{if } V_1 \ll 1 \\ 1 - \frac{\cos^2 \alpha}{K} & \text{if } V_1 \rightarrow 1, \text{ arbitrary } \alpha \\ (1 - \frac{1}{K}) \left(1 + \frac{\sin^2 \alpha}{K} \right) & \text{if } \alpha \rightarrow 0 \end{cases} \quad (5)$$

Consider next a small section of the non-spherical shock at the spherical polar angle θ propagating at an angle

$$\tan \alpha = -\frac{\partial \ln R}{\partial \theta} \quad (6)$$

to the radius vector (Fig. 1). Then

$$\left(\frac{\partial R}{\partial t}\right)^2 = V^2 = \frac{1 - 1/K}{1 - \frac{(\partial_\theta \ln R)^2}{K(1 + (\partial_\theta \ln R)^2)}} \quad (7)$$

Here K is a function of the shock position and angle $K \equiv K(\theta, R)$. Equation (7) is the sought relativistic generalization of the Kompaneets equation (the thin shell modification of the Kompaneets equation, to be more precise). For example, for non-relativistic $V = 1 - 1/K \ll 1$ eq. (7) reduces to the familiar Kompaneets form

$$\frac{\partial R}{\partial t} = V \sqrt{1 + (\partial_\theta \ln R)^2} \quad (8)$$

The other two simplifying cases of the Kompaneets equation include relativistic motion, $K \gg 1$:

$$\left(\frac{\partial R}{\partial t}\right)^2 = 1 - \frac{1}{K} \left(\frac{1}{1 + (\partial_\theta \ln R)^2} \right) \quad (9)$$

and arbitrary quasi-spherical motion, $\alpha \ll 1$, K -arbitrary

$$\left(\frac{\partial R}{\partial t}\right)^2 = \left(1 - \frac{1}{K}\right) \left(1 + \frac{1}{K} (\partial_\theta \ln R)^2\right) \quad (10)$$

Again, the last equation readily gives the standard Kompaneets equation as $K \rightarrow 1$ and $1 - 1/K \rightarrow V^2$.

Examination of the eq. (7) confirms that for relativistic motion the angle $\alpha \sim 1/(\Gamma^2 \Delta\theta) \sim 1/(K \Delta\theta)$ (where $\Delta\theta$ is a typical angular scale for a change in a Lorentz factor. Thus, unless $\Delta\theta \sim 1/\Gamma^2 \sim 1/K$, the term $(\partial_\theta \ln R)^2 \sim 1/K^2$ is of much higher order in $1/K$ and can be neglected. This express the fact the the lateral dynamics of strongly relativistic shock waves (in fact of any strongly relativistic motion) is “frozen out” by kinematic effects.

If the shock is not strongly collimated, $\Delta\theta \gg 1/\Gamma^2$, we can neglect the factor $\partial_\theta \ln R$, different parts of the shock will propagate radially with a different Lorentz factor given by the driver or the external density inhomogeneity:

$$\left(\frac{\partial R(t, \theta)}{\partial t}\right)^2 = 1 - 1/K(t, \theta) \quad (11)$$

This approximation is similar to Laumbach & Probstein (1969) approximation, which assumes radial motion. Thus, in the strongly relativistic case both Kompaneets and Laumbach-Probstein become equivalent, consistent with the conclusion of Shapiro (1979).

In the strongly relativistic regime, $K \equiv \Gamma_1^2 \gg 1$, the lateral dynamics of the flow is frozen out, unless the flow is extremely strongly collimated with $\Delta\theta \leq 1/\Gamma_1^2$. Keeping $(\partial_\theta \ln R)$ arbitrary and expanding in $1/K$ eq. (7) takes the form

$$\left(\frac{\partial R}{\partial t}\right)^2 \sim 1 - \frac{1}{K} \left(\frac{1}{1 + (\partial_\theta \ln R)^2} \right) \quad (12)$$

3. Discussion

In this paper we considered the lateral evolution of non-spherical relativistic outflows. Contrary to the commonly assumed fast lateral expansion, we find that unless the shape of the shock is extremely narrow, with the opening angle of the order of $1/\Gamma^2$, the lateral evolution is effectively frozen out by the highly relativistic motion of the shock. Thus, we confirm the conclusion of Shapiro (1979) that highly relativistic shock propagate nearly ballistically. Our conclusion is broadly consistent with the results of numerical simulations showing very slow (logarithmic, *e.g.*,) lateral evolution. In contrast, the calculations of the afterglow emitted spectra have to be reconsidered accordingly.

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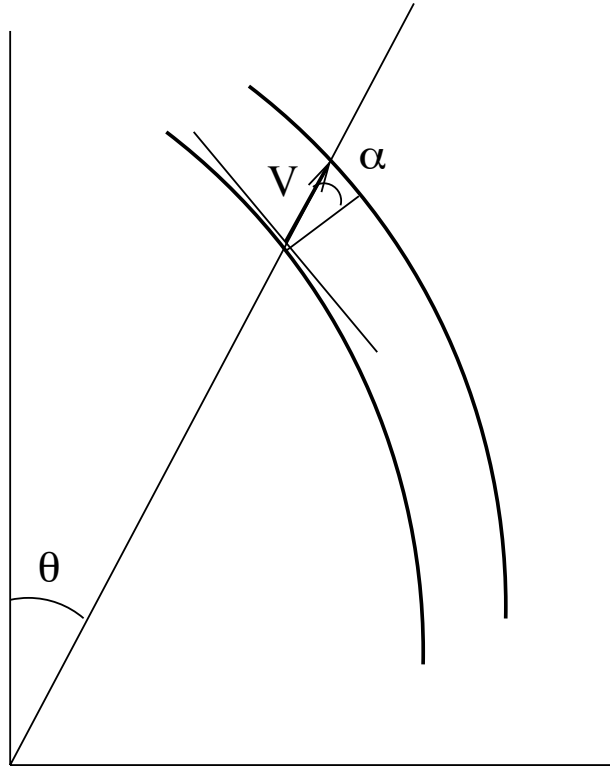


Fig. 1.— Geometry of the flow. The central source located at the origin produces anisotropic wind with luminosity depending on the polar angle θ . At a polar angle θ the shock is located at radius $R(\theta)$, while the wind direction (radial direction) makes an angle α with the shock normal